This follows as a special case of Pascal's Theorem for a hexagon inscribed in a conic.* The conic is in this case degenerate, two straight lines. But (23), (56) are parallel and hence must have an infinitely distant point in common with PQ. Therefore PQ is parallel to AA'.

Also solved by MARJORIE L. BROWN, O. S. ADAMS (two methods), F. E. WOOD, NATHAN ALTSHILLER, HANNAH SUFFIN, and H. H. CONWELL.

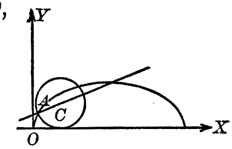
CALCULUS.

416. Proposed by CHARLES N. SCHMALL, New York City.

If A be a point on a cycloid and C the corresponding position of the center of the generating circle, show that AC envelops another cycloid half the size of the first.

SOLUTION BY A. M. HARDING, University of Arkansas.

The coördinates of any point on the cycloid are $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, and the coördinates of the center of the generating circle are given by $x = a\theta$, $y = a$. The equation of AC is



$$\frac{x - a\theta}{\sin\theta} = \frac{y - a}{\cos\theta},$$

or

$$y - a = \cot\theta(x - a\theta).$$

The equation of a cycloid half the size of the given cycloid and having a cusp at O is

$$x = \frac{a\varphi}{2} - \frac{a}{2}\sin\varphi, \qquad y = \frac{a}{2} - \frac{a}{2}\cos\varphi.$$

We propose to show that AC is always tangent to this cycloid.

The equation of any tangent to this cycloid is

$$y - \frac{a}{2}(1 - \cos\varphi) = \cot\frac{\varphi}{2}\left[x - \frac{a}{2}(\varphi - \sin\varphi)\right].$$

Let $\varphi = 2\theta$. This equation then becomes

$$y - a\sin^2\theta = \cot\theta[x - a\theta + a\sin\theta\cos\theta]$$

or

$$y - a = \cot\theta(x - a\theta),$$

which is the same as the equation of AC.

Also solved by C. N. SCHMALL, ELIJAH SWIFT, G. W. HARTWELL, HORACE OLSON, O. S. ADAMS, M. R. GAFFET, R. H. HOWARD, J. B. REYNOLDS, and SHIMPEI NISHIMURA.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated show that $(\sqrt{-1})^{\sqrt{-1}} = 0.207879576351$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

It is not difficult to show, as required in Todhunter's *Plane Trigonometry*, Ed. 1913, pp. 320–21, Examples 266, 275, "that $(a + bi)^{\alpha+\beta i}$ will be wholly real or imaginary if

$$(\beta/2)\log(a^2 + b^2) + \alpha\tan^{-1}(b/a)$$

is (I) zero, or an even multiple of $\pi/2$; or (II) an odd multiple $\pi/2$". In the problem, $a = \alpha = 0$, and $b = \beta = 1$, the conditions corresponding to (I), 0 being called an even number. $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{1}{2}\pi}$, and the numerical value can be tested by using enough decimals in the values of e and π.

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* Cremona, *Elements of Projective Geometry*, translated by Leudensdorf, Articles 88 and 153, 3d edition.

The problem is of some historical interest, being due to John Bernoulli. The distinguished American astronomer, G. W. Hill, proposed it in *The Analyst*, Vol. II, January, 1875, p. 31, Ex. 56, with no reference to its history. It was solved by Walter Siverly, and Henry Heaton in the succeeding number of *The Analyst*, by the following method:

By Euler's Theorem, $\cos\theta + i\sin\theta = e^{i\theta}$, where θ may be $(4n+1)(\pi/2)$. Assume $n = 0$. Then $i = e^{(\pi/2)i}$ and $(i)^i = e^{-\frac{1}{2}\pi} = 0.2078795763507$.

Note.—By an oversight, this problem was placed under Calculus.

Solutions were also received from J. B. Reynolds, Paul Capron, O. S. Adams, and E. B. Escott.

Mr. Escott, using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763491790781.—Editors.

MECHANICS.

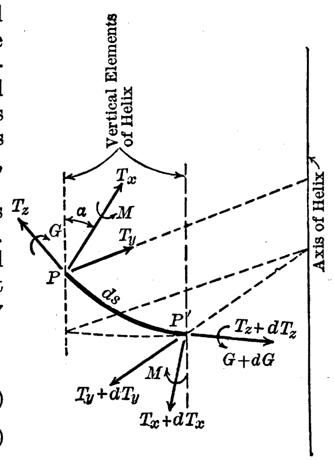
300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of twenty turns of steel wire 0.258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lbs., what is the maximum stress in the coil, its axial compression and its resilence?

Solution by G. Paaswell, N. Y. City.

This solution follows the method given by Love in his treatise on Elasticity.

At any point, P, of the helix, take a system of rectilinear coördinates chosen as follows: the tangent, as the z axis; the radial element as the y axis; and the normal to the two above, as the x axis. The motion of the curve may be analyzed into a curvature k about the x axis described by the z axis and a twist, t, about the z axis described by the x axis. These are, respectively, if the pitch of the helix is a, $\cos^2 a/r$, $(\sin a \cos a)/r$, where r is the radius of the coil. This curvature, k, causes a bending moment, M, and the twist, t, causes a torsional moment, G.



If s is the total length of the helix; n, the number of turns and h the height of the coil; then $h = s\sin a$ and $s\cos a = 2\pi nr$.

Assume the spring subjected to an elongating force R and consider a small free portion of length ds (see figure). The twist from P to P' is tds and the curvature, kds. From ordinary statics,

$$\Sigma_x = \Sigma_y = \Sigma_z = 0,$$

whence

$$T_x - (T_x + dT_x) + tds(T_y + dT_y) = 0, \qquad (1)$$

$$T_y - (T_y + dT_y) - tds(T_x + dT_x) + kds(T_z + dT_z) = 0, \qquad (2)$$

$$T_z - (T_z + dT_z) - kds(T_y + dT_y) = 0, \qquad (3)$$

$$\Sigma_{\text{mom. } x} = \Sigma_{\text{mom. } y} = \Sigma_{\text{mom. } z} = 0,$$

$$(T_y + dT_y)ds = 0;\ \text{whence } T_y = 0 \text{ and } (T_x + dT_x)ds + Gkds - Mtds = 0.$$

From the above, we get at once $T_x - Mt + Gk = 0$ and $tT_x - kT_z = 0$, or $T_x \tan a = T_z$.

The applied loading deforms the helix into a new one with pitch a' and radius r'. The new curvature and twist is, then, t', k'. Assuming that the original curve was under no stress, the bending and torsional moments are given, respectively, by the difference in the curvatures and twists, multiplied by the flexural and torsional rigidities. Then, $M = EI(k' - k)$ and $G = E_sI_s(t' - t)$, where E is Young's Modulus; E_s is the shearing modulus, I is the moment of inertia of the wire about a diameter; and I_s is the polar moment of inertia of the wire. If Poisson's ratio is taken as 1/4, then $E_s = \frac{2}{5}E$. Representing EI by A, then $M = A(k' - k)$ and $G = \frac{4}{5}A(t' - t)$. (6) can now be written $T_x = A\{t'(k' - k) - \frac{4}{5}k'(t' - t)\}$. Since the resultant stress must equal R, $R = T_x \sec a'$. Likewise, the resultant moment is

$$W = M\cos a' + G\sin a'.$$